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On the approximability of the ‘ground-state’ problem for Ising spin glasses by quantum devices

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Abstract. A lower bound result about the approximability of the energy function of Ising spin glasses in the quantum context is determined. The ground-state problem for the three-dimensional Ising spin glasses represented on two-level grids such that the vertical interactions are at most \sqrt{n} (where n is the number of spins set on the grid), is considered. We prove that, unless $\text{NP} \subseteq \text{BQP}$, there is a constant $\alpha > 0$ such that every quantum approximate polynomial-time algorithm finds a solution with absolute error greater than $\alpha\sqrt{n}$, with high probability, infinitely often.

1. Introduction

The study of spin glasses has become of great interest both in solid state physics and in statistical physics. The prototype of a spin glass is a dilute magnetic alloy, for example 1% of magnetic impurities, Mn or Fe, embedded in Cu or Au.

We are concerned with the determination of the ground-state values which minimize the energy function of these disordered systems for which no satisfactory answers have been found.

Bieche *et al* [1] studied the problem for an Ising spin glass represented on a planar lattice and described by the frustration model, for which the spin interactions assume only two symmetrical values, $\pm J$. They provided the exact solutions in polynomial time by using Edmonds’ algorithm for the minimum perfect matching problem.

Later, under the conjecture $\text{P} \neq \text{NP}$, Barahona [2] proved that there is no polynomial-time algorithm that finds the minimum values of the energy function of a three-dimensional Ising spin glass with nearest-neighbour interactions chosen randomly from the set $\{-1, 0, 1\}$. This result forces us to look for algorithms that determine approximate solutions in a polynomial amount of computing time, since correct solutions need super-polynomial time.

Let us consider the simple three-dimensional spin glass model used by Barahona, which is represented by a two-level finite grid whose nodes correspond to the spins and whose edges denote the interactions between nearest-neighbour spins. In particular we consider two-level squared grids (\hat{V}, \hat{E}) with a fixed number of vertical interactions. Each grid is a graph (V, E) such that the set of nodes is $V = \{1, \dots, k\} \times \{1, \dots, k\} \times \{1, 2\}$, for a suitable k , E contains edges whose nodes differ by a component, and the vertical edges $\{(x, y, 1), (x, y, 2)\}$ are at most \sqrt{n} , if $n = 2k^2$ is the cardinality of V . Moreover, if we associate with each node $i \in \hat{V}$ a variable $\sigma_i \in \{-1, 1\}$, representing the spin orientation, and with each edge $\{i, j\} \in \hat{E}$ a weight $J_{ij} \in \{-1, 0, 1\}$, indicating the interaction between

nearest-neighbour spins, we obtain a weighted grid $\hat{G} = (\hat{V}, \hat{E}, J)$. Let \mathcal{G} be the class of the weighted grids just described. The energy of a spin configuration is obtained by the Hamiltonian function $H_{\hat{G}} : \{-1, 1\}^n \rightarrow \mathbb{Z}$ defined for any $\sigma = [\sigma_1, \dots, \sigma_n] \in \{-1, 1\}^n$ by the law

$$H_{\hat{G}}(\sigma) = - \sum_{\{i,j\} \in \hat{E}} J_{ij} \sigma_i \sigma_j.$$

The *ground-states* of our Ising spin glass model are the configurations σ which minimize the function $H_{\hat{G}}(\sigma)$.

The results provided by Bertoni *et al* [3] refer to grids in the class \mathcal{G} . Let n be the number of spins set on the grid; with respect to computation of the ground-states of such a system, they proved that:

- (i) there is an approximate polynomial-time algorithm with absolute error less than \sqrt{n} ;
- (ii) if $P \neq NP$, there exists a positive constant α such that every approximate polynomial-time algorithm has an absolute error greater than $\alpha\sqrt{n}$, infinitely often.

In this study, we will prove that, under the conjecture $NP \not\subseteq BQP$, there exists a constant $\alpha > 0$ such that every quantum approximate polynomial-time algorithm determines, with a probability of error at most $1/4$, a solution with absolute error greater than $\alpha\sqrt{n}$, infinitely often (if n is the number of spins on the grid).

2. Preliminary notions of complexity theory

We introduce some basic notions about computational complexity that we will use to state our results.

A *decision problem* Π can be described as a set of instances D_{Π} and a subset $Y_{\Pi} \subseteq D_{\Pi}$ of instances which give a positive answer. Let Σ be a finite alphabet and Σ^* the set of all words on Σ ; by means of a suitable encoding scheme, a decision problem Π can be represented as a *language* $L \subset \Sigma^*$, that is a set of words which codify the instances of Y_{Π} . A natural measure of the ‘size’ of an instance is the length of the corresponding word represented by the number of symbols composing it.

Turing machines are devices normally used to solve problems (or to recognize languages) consuming computational resources, such as time or space. Usually, a problem is considered ‘practically solvable’ if there is a Turing machine which finds a correct solution by using an amount of time bounded by a polynomial in the size of the input; the class of such problems is denoted by P [4].

The use of randomized algorithms [5] seems to improve the computational power. If BPP represents the class of problems solvable in polynomial time by probabilistic Turing machines with error probability smaller than $1/4$, it is conjectured that there are problems in BPP which are not contained in P , even if no formal proofs have still been founded.

Furthermore, Feynman [6] suggested that a quantum device might potentially be more powerful than a deterministic or randomized Turing machine. This assumption was based on the impossibility for classical devices to simulate the quantum situations without an exponential slowdown.

Feynman’s hypothesis is supported by precise results; for example, in 1994 Shor [7] built an algorithm for quantum Turing machines that solves efficiently the integer factorization problem, whereas it seems that no classical device is able to find a correct solution in polynomial time.

Deutsch [8] gave the first formal description of a quantum Turing machine and we assume the definition contained in [9].

Definition 2.1. A quantum Turing machine (QTM) consists of a finite state set Q , a finite alphabet Σ and a quantum finite-state control

$$\delta: Q \times \Sigma \times \Sigma \times Q \times \{L, R\} \rightarrow \mathbb{C}$$

where, for any $(p, \sigma, \tau, q, d) \in Q \times \Sigma \times \Sigma \times Q \times \{L, R\}$, the complex number $\delta(p, \sigma, \tau, q, d)$ represents the *amplitude* with which the machine in state p reading σ will write τ , enter state q and move in direction d .

The *configuration space* of a classical Turing machine is the following set

$$\Omega = Q \times \Sigma^* \times \Sigma \times \Sigma^*$$

where Q is the finite state set of the machine and $\Sigma^* \times \Sigma \times \Sigma^*$ represents the possible contents of the tape. At any instant the configuration of the machine is an element of a countably infinite set Ω and it can be rewritten as

$$\Omega = \{S_i\}_{i \in \mathbb{N}}$$

following a suitable numeration.

The QTM is, by definition, a quantum mechanical system; we know that the configuration space of a quantum mechanical system is represented by a Hilbert space. What we are looking for, is a Hilbert space capable of containing a countably infinite orthonormal basis that can be put in correspondence with the set Ω .

If \mathcal{H} is a complex separable Hilbert space and $\mathcal{B} \subseteq \mathcal{H}$ is a (countably infinite) orthonormal basis of \mathcal{H} , following Dirac notation, we will denote by $|\cdot\rangle : \Omega \mapsto \mathcal{B}$ the searched bijection and $\mathcal{B} = \{|S_i\rangle\}_{i \in \mathbb{N}}$ is called the *computational basis* [8]. Each unitary element ψ of \mathcal{H} represents a possible configuration of the QTM and it can be written as a linear combination of elements of the computational basis

$$\psi = \sum_{i \in \mathbb{N}} a_i |S_i\rangle$$

where $a_i = \langle \psi | S_i \rangle$ are the *Fourier coefficients* of ψ relative to the chosen computational basis. In the quantum computational context, each a_i is called the *amplitude* of ψ relative to the configuration S_i .

According to the laws of quantum mechanics, the machine transition operator U_δ (induced by the finite state control δ), is a unitary operator in the space \mathcal{H} . The computation of the QTM is the sequence of configurations obtained by the subsequent application of the operator U_δ to the configuration of the machine, starting from an initial configuration.

From a more practical point of view, in general people consider a finite-dimensional Hilbert space \mathcal{H} . If n is the dimension of \mathcal{H} , the basis is $\mathcal{B} = \{S_1, \dots, S_n\}$. Then, in this context, the unitary operator is represented by a square matrix of order n . At a chosen instant, the machine is measured by a suitable observable O , whose spectral representation is just expressed through the fixed basis as follows

$$O = \sum_{i=1}^k \lambda_i |S_i\rangle \langle S_i| \quad (k \leq n).$$

This operation provides the following statistical result: let us suppose that the final configuration of the machine is ψ^f ; then, the probability that the system, measured by the observable O , produces the value S_i (meaning that the machine now occupies the state $|S_i\rangle$) is given by

$$|a_i^f|^2 = |\langle \psi^f | S_i \rangle|^2.$$

In general, a set $\mathcal{S} \subset \Omega$ of configurations is fixed, each representing a successful computation; in this case, the probability of reaching a successful final configuration, through the observable O , is

$$\sum_{S_i \in \mathcal{S}} |\langle \psi^f | S_i \rangle|^2.$$

Bennett [10] and, later, Yao [11] have provided some results that are useful in the construction of quantum algorithms:

(i) a deterministic computation is performable on a quantum machine if and only if it is reversible;

(ii) any polynomial size unitary matrix can be approximated using a polynomial number of elementary unitary transformations; this means that it can be approximated on a quantum machine in polynomial time.

The previous definition of a quantum Turing machine suggests the natural introduction of the following complexity class.

Definition 2.2. [9]: BQP is the class of languages that are accepted by a polynomial-time quantum Turing machine with error probability at most 1/4.

As far as the defined complexity classes, it is conjectured that $P \subset BPP \subset BQP$.

Unfortunately, many problems, of interest from a practical point of view, do not seem to be solvable in polynomial time. An important class involving them is the class of NP-complete problems. NP is the class of problems solvable in polynomial time by a non-deterministic Turing machine [4]; obviously $P \subseteq NP$ and it is conjectured that $P \neq NP$. Likewise, it is improbable that $NP \subseteq BQP$, even if no proof is available.

Intuitively the NP-complete problems are ‘the most difficult’ in the class NP, in the sense that building a polynomial-time algorithm solving one of them, would imply that $P = NP$. We are interested in this kind of problem because the decision version of the ground-state problem is NP-complete [2].

In this study we accept the conjecture that $NP \not\subseteq BQP$ and therefore that NP-complete problems cannot be solvable in polynomial time by any quantum Turing machine.

3. Combinatorial optimization problems and their approximability

First of all, we introduce some formal notions about combinatorial optimization theory.

Definition 3.1. [12]: An *optimization problem* Π is defined by the tuple $\langle \text{In}_\Pi, \text{Sol}, \omega, \text{opt} \rangle$ such that:

- (i) In_Π represents the set of the instances of Π ;
- (ii) given an instance $I \in \text{In}_\Pi$, $\text{Sol}(I)$ denotes the set of its feasible solutions; this set is recognizable in polynomial time and there is a polynomial p such that for any $S \in \text{Sol}(I)$, $|S| \leq p(|I|)$;
- (iii) ω is the *objective function* that associates a non-negative rational number $\omega(I, S)$ (the *solution value*) to any couple (S, I) ; in the sequel, given the instance I , we will denote $\omega^*(I)$ as the measure of an optimum solution of the problem Π ;
- (iv) $\text{opt} \in \{\max, \min\}$ explains if Π is a maximization or a minimization problem.

As an example, a well-known optimization problem is MAX 3SAT [12]:

• **MAX 3SAT**

Instance: A set V of Boolean variables, a collection C of clauses over V such that each of them has three literals at most.

Solution: A truth assignment to the variables in V .

Solution value: The number of satisfied clauses.

Opt: Max.

Combinatorial optimization problems admit a natural version as decision problems (i.e. problems with a *yes-no* answer); here we give two examples:

• **MAX CUT-3**

Instance: A graph $G = (V, E)$ of degree $D \leq 3$ and an integer k .

Question: Is there a subset $V_1 \subseteq V$ such that the cardinality of $\text{cut}(V_1)$ is greater than k , where $\text{cut}(V_1)$ is the set of edges with one endpoint in V_1 and the other one in $V - V_1$?

• **MAX $\{-1, 0, 1\}$ -CUT**

Instance: A weighted grid $\hat{G} = (\hat{V}, \hat{E}, J) \in \mathcal{G}$ and an integer k .

Question: Is there a subset $\hat{V}_1 \subseteq \hat{V}$ such that the weight $W(\text{cut}(\hat{V}_1))$ is greater than k , where $W(\text{cut}(\hat{V}_1)) = \sum_{\{i,j\} \in \text{cut}(\hat{V}_1)} J_{ij}$?

The ground-state problem may be stated as a minimization problem in the following way:

• **GROUNDSTATE (GS)**

Instance: A weighted grid $\hat{G} = (\hat{V}, \hat{E}, J) \in \mathcal{G}$.

Question: Find a configuration of spins $\sigma = [\sigma_1, \dots, \sigma_n]$ that minimizes the expression

$$H_{\hat{G}}(\sigma) = - \sum_{\{i,j\} \in \hat{E}} J_{ij} \sigma_i \sigma_j.$$

If the decision version of an optimization problem is NP-complete, then it is impossible to obtain the solution in polynomial time, unless $P = NP$. Nevertheless, we can hope to find a 'good approximate' solution by means of a polynomial-time 'approximate' algorithm.

Two measures of the quality of a solution result from the absolute and relative error.

Definition 3.2. Given an optimization problem Π , an instance $I \in \text{In}_{\Pi}$ and a solution $S \in \text{Sol}(I)$, the *absolute error* $E(I, S)$ and the *relative error* $\text{Err}(I, S)$ are respectively

$$E(I, S) = |\omega^*(I) - \omega(I, S)| \quad \text{Err}(I, S) = \frac{|\omega^*(I) - \omega(I, S)|}{\max\{\omega^*(I), \omega(I, S)\}}.$$

Remark 1. There exists a relation between the ground-state problem and MAX $\{-1, 0, 1\}$ -CUT based on the absolute error for an approximate algorithm applied to the instances of the two problems.

Let $\hat{G} = (\hat{V}, \hat{E}, J)$ be an instance of GS, $(\hat{V}, \hat{E}, -J)$ the instance of MAX $\{-1, 0, 1\}$ -CUT and A an approximate polynomial-time algorithm. This algorithm will determine a configuration σ_0 for GS and a subset $\hat{V}_1 \subseteq \hat{V}$ for MAX $\{-1, 0, 1\}$ -CUT. Then it is possible to express a relation between the energy $H_{\hat{G}}(\sigma_0)$ of the system in the configuration σ_0 and the maximum weight W of $\text{cut}(\hat{V}_1)$ in this way

$$H_{\hat{G}}(\sigma_0) = - \sum_{\{i,j\} \in \hat{E}} J_{ij} - 2W$$

that is

$$W = \frac{1}{2} \left(- \sum_{\{i,j\} \in \hat{E}} J_{ij} - H_{\hat{C}}(\sigma_0) \right).$$

Through these expressions, it is easy to verify that an approximate polynomial-time algorithm for GS with absolute error E is an approximate polynomial-time algorithm for MAX $\{-1, 0, 1\}$ -CUT with absolute error $E/2$ and vice versa.

The notion of a polynomial-time approximation scheme ensures that we will find ‘arbitrarily good’ approximate solutions in polynomial time.

Definition 3.3. An optimization problem Π admits a *polynomial-time approximation scheme* (PTAS) if, for every $\varepsilon > 0$, there exists an approximate polynomial-time algorithm A_ε such that, for every instance I , the solution $A_\varepsilon(I)$ satisfies the following relation

$$\text{Err}(I, A_\varepsilon(I)) \leq \varepsilon.$$

The class of problems that admit a polynomial-time algorithm with error smaller than one is called SNP [13]. The ‘most difficult to solve’ problems in SNP are said *SNP-complete* (for a formal definition see [13]); here ‘most difficult’ means that, if an SNP-complete problem admits a PTAS, the same holds for any problem in SNP. Two well-known SNP-complete problems are MAX 3SAT and MAX CUT-3.

An important result in this area shows that SNP-complete problems do not admit a PTAS.

Theorem 3.1. [14]: If $P \neq NP$, MAX 3SAT does not admit a PTAS.

Maximization problems in SNP are called MAX SNP problems. From theorem 3.1, it follows that MAX SNP-complete problems are not solvable by any PTAS [13].

Now we discuss the combinatorial optimization notions from a quantum point of view.

Definition 3.4. Let Π be an optimization problem and A a quantum approximate polynomial-time algorithm. The *absolute error* associated with A is smaller than $E(n)$ if, with probability at least $3/4$, on input of size n , A outputs an approximate solution with absolute error less than $E(n)$. Likewise, A has *relative error* less than $\varepsilon > 0$ if, for any $I \in \text{In}_\Pi$

$$\sum_{\text{Err}(I,S) \leq \varepsilon} |\lambda_{I,S}|^2 \geq \frac{3}{4}$$

where $S \in \text{Sol}(I)$ and $\lambda_{I,S}$ represents the amplitude associated with the quantum computation that gives S as output when I is the input.

Definition 3.5. An optimization problem Π is solvable by a *quantum polynomial-time approximation scheme* (QPTAS) if, for every $\varepsilon > 0$, there exists a quantum approximate polynomial-time algorithm A_ε that, for any instance $I \in \text{In}_\Pi$, finds a solution S with relative error smaller than ε .

Through the notion of PTAS, we can state the following proposition, since it already holds in the classical context.

Proposition 3.1. Let Π be an optimization problem that belongs to the class MAX SNP. If Π admits no QPTAS, then any MAX SNP-complete problem is not solvable by any QPTAS.

4. Results

In this section we consider Ising spin glasses represented on two-level weighted grids from the class \mathcal{G} . We prove a lower bound result about the approximability of the energy function of such physical systems by using the quantum notions of computational complexity and the optimization combinatorial theory. Since this result is a lower bound result, it holds for more general spin glass models. In this context the hypothesis $P \neq NP$ is replaced by the conjecture $NP \not\subseteq BQP$.

The first step towards obtaining our results consists in proving the following theorem.

Theorem 4.1. Let us consider an NP-complete problem Π and a reduction to the problem MAX 3SAT, $f: \text{In}_\Pi \rightarrow \text{In}_{\text{MAX 3SAT}}$. Let us suppose that

$$\omega^*(f(I)) = \begin{cases} c & \text{if } I \in Y_\Pi \\ (1-g)c & \text{if } I \notin Y_\Pi \end{cases}$$

where $g < 1$ is a positive constant and c is the number of clauses in $f(I)$. Then, if $NP \not\subseteq BQP$, MAX 3SAT does not admit a QPTAS.

Proof. Let us suppose that MAX 3SAT is solvable by a QPTAS; then we denote by $A_{g/3}$ the quantum approximate polynomial-time algorithm that finds an approximate solution for the problem MAX 3SAT with relative error less than $\varepsilon = g/3$ (with 3/4 confidence level). Let us consider the following algorithm:

Algorithm B_Π :

input: An instance $I \in \text{In}_\Pi$;

(1) compute $f(I)$;

(2) determine a solution S with probability $|\lambda_{f(I),|S}|^2$ by applying $A_{g/3}$ to $f(I)$;

(3) **if** $\omega(f(I), S) \geq (1-g/3)$

then output 'yes',

else output 'no'.

B_Π is a quantum approximate polynomial-time algorithm that solves Π ; in fact: when $I \in Y_\Pi$, $B_{g/3}$ outputs 'yes' with probability at least 3/4, since, with the same probability, it holds that

$$|\omega(f(I), S) - c| \leq cg/3$$

that is

$$\omega(f(I), S) \geq c(1-g/3).$$

On the other hand, when $I \notin Y_\Pi$, with probability greater than 3/4, $B_{g/3}$ outputs 'no' since

$$\omega(f(I), S) \leq \omega^*(f(I)) = c(1-g) < c(1-g/3).$$

Since Π is NP-complete, that implies $NP \subseteq BQP$. □

As MAX 3SAT belongs to the class MAX SNP, from the just proved theorem and from proposition 3.1, a property about MAX SNP-complete problems is derived.

Theorem 4.2. If $NP \not\subseteq BQP$, MAX SNP-complete problems are not solvable by any QPTAS.

Now we consider the MAX CUT problem for which instances are connected graphs.

Theorem 4.3. If $\text{NP} \not\subseteq \text{BQP}$, MAX CUT restricted to connected graphs does not admit any QPTAS.

Proof. Let A_ε be the quantum approximate polynomial-time algorithm for MAX CUT restricted to connected graphs with relative error ε . Consider the following algorithm.

Algorithm B_ε :

input: An arbitrary graph $G = (V, E)$ (let $|V| = n$);

(1) determine the connected components $G_1 = (V_1, E_1), \dots, G_l = (V_l, E_l)$ of G ;

(2) **for** $i = 1, \dots, l$ **do** find $s = \lceil \log n \rceil$ solutions W_{1i}, \dots, W_{si} by applying s times A_ε to the connected component G_i ; let W_i be the element in $\{W_{1i}, \dots, W_{si}\}$ with maximum cut;

output: $W = W_1 \cup \dots \cup W_l$.

B_ε is a quantum approximate polynomial-time algorithm for MAX CUT. In fact the time complexity of B_ε is bounded by the quantity $\log nl P_\varepsilon$, where P_ε is the time complexity of A_ε and l is the number of connected components of the input.

Moreover it holds that: $\sum_{\text{Err}(G, W) \leq \varepsilon} |\lambda_{G, |W|}|^2 \geq \frac{3}{4}$. With regard to this fact, we first observe that $\text{Err}(G_i, W_i) \leq \varepsilon$ for all i implies that $\text{Err}(G, W) \leq \varepsilon$. Hence

$$\sum_{\text{Err}(G, W) \leq \varepsilon} |\lambda_{G, |W|}|^2 \geq \sum_{\bigwedge_{i=1}^l \text{Err}(G_i, W_i) \leq \varepsilon} |\lambda_{G, |W_1|, \dots, |W_l|}|^2.$$

Because of the independent applications of the algorithm A_ε to the connected components, we obtain

$$\begin{aligned} \sum_{\bigwedge_{i=1}^l \text{Err}(G_i, W_i) \leq \varepsilon} |\lambda_{G, |W_1|, \dots, |W_l|}|^2 &= \sum_{\bigwedge_{i=1}^l \text{Err}(G_i, W_i) \leq \varepsilon} |\lambda_{G_1, |W_1|}|^2 \cdots |\lambda_{G_l, |W_l|}|^2 \\ &= \prod_{i=1}^l \left(1 - \prod_{k=1}^s \sum_{\text{Err}(G_i, W_{ki}) > \varepsilon} |\lambda_{G_i, |W_{ki}|}|^2 \right) \\ &\geq \left(1 - \left(\frac{1}{4} \right)^s \right)^l \geq \left(1 - \left(\frac{1}{4} \right)^s \right)^n \geq \frac{3}{4}. \end{aligned}$$

In this way we have built the QPTAS $\{B_\varepsilon\}$ that solves the MAX CUT problem. As MAX CUT is a MAX SNP-complete problem, we have proved a result in contradiction with theorem 4.2. \square

The theorem just demonstrated is true even when we deal with graphs of degree at most three, since the decision version of the MAX CUT problem for graphs with degree at most three is still NP-complete.

Corollary 1. If $\text{NP} \not\subseteq \text{BQP}$, there exists no QPTAS that solves MAX CUT for which instances are connected graphs of degree at most three.

Finally, we state our main result. If \hat{G} is a grid in the class \mathcal{G} and A is a quantum approximate polynomial-time algorithm, we will indicate by $H(A(\hat{G}))$ the energy of the spin glass in the configuration determined by A and with $H^*(\hat{G})$ the minimum energy of the system.

Theorem 4.4. If $\text{NP} \not\subseteq \text{BQP}$, there exists a positive constant α such that, for every quantum approximate polynomial-time algorithm A , there are infinitely many grids $\hat{G} = (\hat{V}, \hat{E}, J)$ in the class \mathcal{G} that satisfy the following relation

$$\sum_{|H^*(\hat{G}) - H(A(\hat{G}))| \geq \alpha \sqrt{|\hat{V}|}} |\lambda_{\hat{G}, |A(\hat{G})|}|^2 \geq \frac{3}{4}.$$

Proof. In contradiction to this thesis, let us suppose that for every constant $\alpha > 0$ there exists a quantum approximate polynomial-time A_α such that

$$\sum_{|H^*(\hat{G}) - H(A_\alpha(\hat{G}))| < \alpha \sqrt{|\hat{V}|}} |\lambda_{\hat{G}, |A_\alpha(\hat{G})|}|^2 \geq \frac{3}{4}$$

but for a finite number of grids \hat{G} from the class \mathcal{G} . In [3] it is proved that there is a polynomial-time computable function f that associates with every instance $G = (V, E)$ of MAX CUT-3 the instance $f(G) = (\hat{V}, \hat{E}, J)$ of MAX $\{-1, 0, 1\}$ -CUT such that

$$\omega^*(G) = W^*(f(G))$$

where ω^* and W^* are the values of the best solutions of MAX CUT-3 and MAX $\{-1, 0, 1\}$ -CUT respectively. Besides, there exists another polynomial-time computable function g that assigns to every instance $G = (V, E)$ of MAX CUT-3 and to every feasible solution $\hat{S} \in \text{Sol}(f(G))$ of MAX $\{-1, 0, 1\}$ -CUT a solution $g(\hat{S}) \in \text{Sol}(G)$ such that

$$W(f(G), S) \leq \omega(G, g(\hat{S})).$$

Consider the following algorithm.

Algorithm \bar{A}_α :

input: A graph $G = (V, E)$ of degree $D \leq 3$

(1) Determine the weighted grid $f(G) = (\hat{V}, \hat{E}, J)$

(2) Apply A_α to the weighted grid $\hat{G} = (\hat{V}, \hat{E}, -J)$

output: $g(A_\alpha(\hat{G})) = \bar{A}_\alpha(G)$.

From remark 1 and from the relation between MAX $\{-1, 0, 1\}$ -CUT and MAX CUT-3

$$|\omega^*(G) - \omega(\bar{A}_\alpha(G))| \leq |H^*(\hat{G}) - H(A_\alpha(\hat{G}))| < \alpha \sqrt{|\hat{V}|}.$$

Therefore, the relative error \bar{A}_α verifies

$$\frac{|\omega^*(G) - \omega(\bar{A}_\alpha(G))|}{\omega^*(G)} \leq \frac{|H^*(\hat{G}) - H(A_\alpha(\hat{G}))|}{\omega^*(G)} < \frac{\alpha \sqrt{|\hat{V}|}}{\omega^*(G)}.$$

A construction provided by Barahona [2] allows one to embed a graph (V, E) of degree $D \leq 3$ into a two-level squared grid (\hat{V}, \hat{E}) such that $|\hat{V}| = 18|V|^2$. Recalling that the cardinality of the maximum cut for a connected graph is $|V| - 1$ at least, we have

$$\text{Err}(G, \bar{A}_\alpha(G)) < \frac{\alpha \sqrt{|\hat{V}|}}{\omega^*(G)} < \frac{\alpha \sqrt{|\hat{V}|}}{|V|} = \alpha \sqrt{18}.$$

We can conclude that

$$\sum_{\text{Err}(G, \bar{A}_\alpha(G)) < \alpha \sqrt{18}} |\lambda_{G, |\bar{A}_\alpha(G)}|^2 \geq \sum_{|H^*(\hat{G}) - H(A_\alpha(\hat{G}))| < \alpha \sqrt{|\hat{V}|}} |\lambda_{\hat{G}, |H(A_\alpha(\hat{G}))|}|^2 \geq \frac{3}{4}.$$

The quantum polynomial-time approximation scheme $\{\bar{A}_\alpha\}$ solves the MAX CUT-3 problem restricted to connected graphs, in contradiction with corollary 1.

This lower bound result proves that, under the conjecture $\text{NP} \not\subseteq \text{BQP}$, algorithms for quantum Turing machines do not improve the classical approximation of the energy function for three-dimensional Ising spin glasses.

5. Conclusions

We can summarize the computational complexity results about the ground-state problem for a three-dimensional Ising spin glass represented on a two-level grid such that, if n is the number of spins set on the grid, the vertical interactions are at most \sqrt{n} .

Barahona proved that GS is an NP-complete problem. This means that, if $\text{P} \neq \text{NP}$, it is not possible to build a polynomial-time algorithm which finds the correct solution.

Besides, we have verified a difficulty in finding good approximate algorithms. In the deterministic case, Bertoni, Campadelli and Molteni have determined a lower bound for the absolute error of any approximate polynomial-time algorithm, unless $\text{P} = \text{NP}$. In this study, under the conjecture $\text{NP} \not\subseteq \text{BQP}$, we have proved that quantum Turing machines are not sufficient to improve the lower bound result $\Omega(\sqrt{n})$.

It would be interesting to look for good approximate algorithms for more general models, for instance when the number of vertical interactions is not bounded or when there are connections among spins which are not necessarily nearest neighbours.

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References

- [1] Bieche I, Maynard R, Rammal R and Uhry J P 1980 On the ground states of the frustration model of a spin glass by a matching method of graph theory *J. Phys. A: Math. Gen.* **13** 2553–76
- [2] Barahona F 1982 On the computational complexity of Ising spin glass models *J. Phys. A: Math. Gen.* **15** 3241–53
- [3] Bertoni A, Campadelli P and Molteni G 1994 On the approximability of the energy function of Ising spin glasses *J. Phys. A: Math. Gen.* **27** 6719–29
- [4] Garey M R and Johnson D S 1979 *Computers and Intractability: a Guide to the Theory of NP-completeness* (San Francisco, CA: Freeman)
- [5] Balcazar J L, Diaz J and Gabarro J 1988 *Structural Complexity I* (New York: Springer)
- [6] Feynman R 1982 Simulating physics with computers *Int. J. Theor. Phys.* **21** 467–88
- [7] Shor P W 1994 Algorithms for quantum computation: discrete log and factoring *Proc. 35th Ann. Symp. on Foundations of Computer Science* ed S Goldwasser (IEEE) pp 124–34
- [8] Deutsch D 1985 Quantum theory, the Church–Turing principle and the universal quantum computer *Proc. R. Soc. Lond. A* **400** 97–117
- [9] Bernstein E and Vazirani U 1993 Quantum complexity theory *Proc. 25th Ann. ACM Symp. on Theory of Computing* pp 11–20
- [10] Bennett C H 1973 Logical reversibility of computation *IBM J. Res. Develop.* **17** 525–32
- [11] Yao A 1993 Quantum circuit complexity *Proc. 34th Ann. Symp. on Foundations of Computer Science* (IEEE) pp 352–61
- [12] Ausiello G, Crescenzi P and Protasi M 1993 Approximate solution of NP-hard optimization problems: a survey
- [13] Papadimitriou C H and Yannakakis M 1991 Optimization, approximation, and complexity classes *J. Comput. Syst. Sci.* **43** 425–40
- [14] Arora S, Lund C, Motwani R, Sudan M and Szegedy M 1992 Proof verification and hardness of approximation problems *Proc. 33rd Ann. Symp. on Foundations of Computer Science* (IEEE) pp 14–23